

# Diffeomorphisms with various $C^1$ –(generic-)stable properties

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## Abstract

Let  $M$  be a smooth compact manifold and  $\Lambda$  be a compact invariant set. In this paper we prove that for every robustly transitive set  $\Lambda$ ,  $f|_{\Lambda}$  satisfies a  $C^1$ –generic-stable shadowable property (resp.,  $C^1$ –generic-stable transitive specification property or  $C^1$ –generic-stable barycenter property) if and only if  $\Lambda$  is a hyperbolic basic set. In particular,  $f|_{\Lambda}$  satisfies a  $C^1$ –stable shadowable property (resp.,  $C^1$ –stable transitive specification property or  $C^1$ –stable barycenter property) if and only if  $\Lambda$  is a hyperbolic basic set.

## 1 Introduction

In the studies of dynamical systems, the pseudo-orbit shadowing property usually plays an important role in the investigation of stability theory and ergodic theory. Wen, Gan and Wen [13] proved that  $C^1$ -stably shadowable chain component is hyperbolic. Lee, Morivasu, Sakai[6] showed that a chain recurrent set has  $C^1$ –stable shadowing property if and only if the system satisfies both Axiom A and the no-cycle and also proved that a chain component containing a hyperbolic periodic point  $p$  has  $C^1$ –stable shadowing property if and only if it is the hyperbolic homoclinic class of  $p$ . Moreover, Tajbakhsh and Lee[12] proved that a homoclinic class has  $C^1$ –stable shadowing property if and only

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if it is hyperbolic. Recently, Sakai, Sumi and Yamamoto showed that a closed invariant set satisfies  $C^1$ –stable specification property if and only if it is a hyperbolic elementary set. Since the specification property there naturally implies topologically mixing, they gave a characterization of the hyperbolic (mixing) elementary sets. Specification property is due to Bowen and Sigmund and holds for every mixing compact set with shadowing property. In the present paper we show that for every transitive compact set with shadowing property, a version of transitive specification property is true. Furthermore, we also discuss a notion called barycenter property due to Abdenur, Bonatti, Crovisier[1], weaker than transitive specification property. Here we are mainly to characterize the diffeomorphisms satisfying  $C^1$ –generic-stable shadowable property, transitive specification property or barycenter property(for particular case,  $C^1$ –stable shadowable property, transitive specification property or barycenter property). More precisely, for a robustly transitive set, it has one of above properties if and only if it is a hyperbolic basic set.

Let  $(M, d)$  denote a compact metric space and let  $f : M \rightarrow M$  be a homeomorphism. Let  $\Lambda$  be a compact and  $f$ –invariant set and let  $f|_\Lambda$  be the restriction of  $f$  on the set  $\Lambda$ . Now we start to introduce the notions of shadowing, specification and barycenter properties. A sequence  $\{y_n\}_{n=a}^b \subseteq \Lambda$  is called a  $\delta$ -pseudo-orbit ( $\delta \geq 0$ ) of  $f$  if  $d(f(y_n), y_{n+1}) \leq \delta$  for every  $a \leq n \leq b$ . A system  $f|_\Lambda$  is said to have the *shadowing property* if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that for a given  $\delta$ -pseudo-orbit  $y = \{y_n\}_{n=a}^b \subseteq \Lambda$ , we can find  $x \in \Lambda$ , which  $\varepsilon$ -traces  $y$ , i.e.,  $d(f^n(x), y_n) < \varepsilon$  for every  $a \leq n \leq b$ .  $f|_\Lambda$  is said to satisfy the *transitive specification property* if the following holds: for any  $\varepsilon > 0$  there exists an increasing sequence of integers  $M_0(\varepsilon) = 0 < M_1(\varepsilon) < M_2(\varepsilon) < \dots$  towards to  $+\infty$  such that for any  $k \geq 1$ ,  $n \geq 1$ , any  $k$  points  $x_1, x_2, \dots, x_k \in \Lambda$ , and any integers  $n_1, n_2, \dots, n_k$ , there exists a point  $z \in \Lambda$  and a sequence of integers  $c_1 = 0 < c_2 < \dots < c_k$  with  $c_{j+1} - c_j - n_j \in [M_{n-1}(\varepsilon), M_n(\varepsilon)]$  ( $j = 1, 2, \dots, k-1$ ) such that  $d(f^{c_j+i}(z), f^i(x_j)) < \varepsilon$ ,  $0 \leq i \leq n_j$ ,  $1 \leq j \leq k$ . Now we begin to recall the barycenter property(a little difference to [1]). Let  $P(f|_\Lambda)$  be the set of periodic points of  $f$  in  $\Lambda$ . In particular, set  $P(f) = P(f|_M)$ . Given two periodic points  $p, q \in P(f|_\Lambda)$ , we say  $p, q$  have the barycenter property, if for any  $\varepsilon > 0$  there exists an integer  $N = M(\varepsilon, p, q) > 0$  such that for any two integers  $n_1, n_2$ , there exists a point  $z \in \Lambda$  and an integer  $X \in [0, N]$  such that  $d(f^i(z), f^i(p)) < \varepsilon$ ,  $-n_1 \leq i \leq 0$ , and  $d(f^{i+X}(z), f^i(q)) < \varepsilon$ ,  $0 \leq i \leq n_2$ .  $f|_\Lambda$  is said to satisfy the *barycenter property* if the barycenter property holds for any two periodic points  $p, q \in P(f|_\Lambda)$ .

Obviously the barycenter property is weaker than the transitive specification property. The transitive specification property means that whenever there are  $k$  pieces of orbits they may be approximated up to  $\varepsilon$  by one orbit, provided that the time for *switching* from the forward piece of orbit to the afterward and the time for *switching back* are bounded between two integers  $M_{n-1}(\varepsilon) \leq M_n(\varepsilon)$ , these integers  $M_n(\varepsilon)$  being independent of the length of the  $k$  pieces of orbits. Here this notion of *transitive specification property* is weaker than the usual *(mixing) specification property* defined by Sigmund[11].

Let  $\Lambda$  be as before.  $\Lambda$  is transitive if there is some  $x \in \Lambda$  whose forward orbit is dense in  $\Lambda$ . A transitive set  $\Lambda$  is trivial if it consists of a periodic orbit. Note that transitive specification property implies that  $\Lambda$  is topologically transitive.  $\Lambda$  is locally maximal

in some neighborhood  $U \subseteq M$  of  $\Lambda$  if  $\Lambda = \bigcap_{k \in \mathbb{Z}} f^k(U)$ . A set  $\Lambda$  is a basic set (resp. elementary set) if  $\Lambda$  is locally maximal and  $f|_{\Lambda}$  is transitive (resp. topologically mixing). In a Baire space  $X$ , we call  $\mathcal{R} \subseteq X$  be a residual set, if it contains a dense  $G_{\delta}$  set.

Let  $M$  be a closed  $C^{\infty}$  manifold and let  $\text{Diff}(M)$  be the space of diffeomorphisms of  $M$  endowed with the  $C^1$ -topology. Denote by  $d$  the distance on  $M$  induced from a Riemannian metric on the tangent bundle  $TM$ . Given  $f \in \text{Diff}(M)$ , denote by  $\mathcal{O}(p)$  the periodic  $f$ -orbit of  $p \in P(f)$ . If  $p \in P(f)$  is a hyperbolic saddle with period  $\pi(p) > 0$ , then there are the local stable manifold  $W_{\varepsilon}^s(p)$  and the local unstable manifold  $W_{\varepsilon}^u(p)$  of  $p$  for some  $\varepsilon = \varepsilon(p) > 0$ . It is easy to see that if  $d(f^n(x), f^n(p)) \leq \varepsilon$  for any  $n \geq 0$ , then  $x \in W_{\varepsilon}^s(p)$  (a similar property also holds for local unstable manifold  $W_{\varepsilon}^u(p)$  with respect to  $f^{-1}$ ). The stable manifold  $W_{\varepsilon}^s(p)$  and the unstable manifold  $W_{\varepsilon}^u(p)$  of  $p$  are defined as usual. The dimension of the stable manifold  $W_{\varepsilon}^s(p)$  is called the index of  $p$ , and denoted by  $\text{index}(p)$ .

An  $f$  invariant compact set  $\Lambda$  is robustly transitive in some neighborhood  $U$  if  $\Lambda$  is locally maximal in  $U$  and there is a  $C^1$ -neighborhood  $\mathcal{U}(f)$  of  $f$  such that for any  $g \in \mathcal{U}(f)$ ,  $\Lambda_g(U) := \bigcap_{k \in \mathbb{Z}} g^k(U)$  (called a continuation of  $\Lambda_f(U) = \Lambda$ ) is transitive.

Now we state our main result as follows.

**Theorem 1.1.** *Let  $\Lambda$  be an  $f$  invariant compact set and assume that  $\Lambda$  is robustly transitive in some neighborhood  $U$ . Then the following conditions are equivalent:*

(1)  $f|_{\Lambda}$  is  $C^1$ -generic-stably shadowable, i.e., there is a  $C^1$ -neighborhood  $\mathcal{U}(f)$  of  $f$  and a residual set  $\mathcal{R} \subseteq \mathcal{U}(f)$  such that for any  $g \in \mathcal{R}$ ,  $\Lambda_g(U) := \bigcap_{k \in \mathbb{Z}} g^k(U)$  has shadowing property;

(1')  $f|_{\Lambda}$  is  $C^1$ -stably shadowable, i.e., there is a  $C^1$ -neighborhood  $\mathcal{U}(f)$  of  $f$  such that for any  $g \in \mathcal{U}(f)$ ,  $\Lambda_g(U) := \bigcap_{k \in \mathbb{Z}} g^k(U)$  has shadowing property;

(2)  $f|_{\Lambda}$  satisfies the  $C^1$ -generic-stable transitive specification property, i.e., there is a  $C^1$ -neighborhood  $\mathcal{U}(f)$  of  $f$  and a residual set  $\mathcal{R} \subseteq \mathcal{U}(f)$  such that for any  $g \in \mathcal{R}$ ,  $\Lambda_g(U) := \bigcap_{k \in \mathbb{Z}} g^k(U)$  has transitive specification property;

(2')  $f|_{\Lambda}$  satisfies the  $C^1$ -stable transitive specification property, i.e., there is a  $C^1$ -neighborhood  $\mathcal{U}(f)$  of  $f$  such that for any  $g \in \mathcal{U}(f)$ ,  $\Lambda_g(U) := \bigcap_{k \in \mathbb{Z}} g^k(U)$  has transitive specification property;

(3)  $f|_{\Lambda}$  satisfies the  $C^1$ -generic-stable barycenter property, i.e., there is a  $C^1$ -neighborhood  $\mathcal{U}(f)$  of  $f$  and a residual set  $\mathcal{R} \subseteq \mathcal{U}(f)$  such that for any  $g \in \mathcal{R}$ ,  $\Lambda_g(U) := \bigcap_{k \in \mathbb{Z}} g^k(U)$  has barycenter property;

(3')  $f|_{\Lambda}$  satisfies the  $C^1$ -stable barycenter property, i.e., there is a  $C^1$ -neighborhood  $\mathcal{U}(f)$  of  $f$  such that for any  $g \in \mathcal{U}(f)$ ,  $\Lambda_g(U) := \bigcap_{k \in \mathbb{Z}} g^k(U)$  has barycenter property;

(4) there is a  $C^1$ -neighborhood  $\mathcal{U}(f)$  of  $f$  and a residual set  $\mathcal{R} \subseteq \mathcal{U}(f)$  such that for any  $g \in \mathcal{R}$ , any two periodic hyperbolic saddles  $p, q \in \Lambda_g(U) := \bigcap_{k \in \mathbb{Z}} g^k(U)$ ,  $W^s(\mathcal{O}(p)) \cap W^u(\mathcal{O}(q)) \neq \emptyset$ ;

(4') there is a  $C^1$ -neighborhood  $\mathcal{U}(f)$  of  $f$  such that for any  $g \in \mathcal{U}(f)$ , any two periodic hyperbolic saddles  $p, q \in \Lambda_g(U) := \bigcap_{k \in \mathbb{Z}} g^k(U)$ ,  $W^s(\mathcal{O}(p)) \cap W^u(\mathcal{O}(q)) \neq \emptyset$ ;

(5) there is a  $C^1$ -neighborhood  $\mathcal{U}(f)$  of  $f$  such that for any  $g \in \mathcal{U}(f)$ , any periodic point of  $\Lambda_g(U) := \bigcap_{k \in \mathbb{Z}} g^k(U)$  is hyperbolic and has the same index;

(6) there is a  $C^1$ -neighborhood  $\mathcal{U}(f)$  of  $f$  such that for any  $g \in \mathcal{U}(f)$ ,  $\Lambda_g(U) := \bigcap_{k \in \mathbb{Z}} g^k(U)$  is a hyperbolic basic set.

**Remarks.** 1. The result in [10] is a particular case of (2') in Theorem 1.1 (being as the mixing case), since the specification property assumed on  $\Lambda_g(U)$  in [10] naturally implies that  $\Lambda_g(U)$  is topologically mixing(i.e.,  $\Lambda$  is robustly mixing). And we point out that the codition (2) in Theorem 1.1 can be also as a generalization of the result[10], since (2) is weaker than (2').

2. From [10] the set of transitive Anosov diffeomorphisms is a characterization of the set of diffeomorphisms satisfying  $C^1$ –stable mixing specification property. Here by Theorem 1.1 it is also a characterization of the set of diffeomorphisms satisfying  $C^1$ –stable (or generic-stable) transitive specification property. Furthermore, if  $M$  is topologically transitive, the set of Anosov diffeomorphisms is also a characterization of  $C^1$ –stable (or generic-stable) shadowable property or  $C^1$ –stable (or generic-stable) barycenter property.

The equivalence of (5) and (6) is due to Mañé [7] and by using this Sakai, Sumi and Yamamoto [10] proved that  $f|_{\Lambda_f(U)}$  satisfies the  $C^1$ –stable specification property (mixing case) if and only if  $\Lambda$  is a hyperbolic elementary set. Actually, it is essentially proved  $(4') \Rightarrow (5)$  in [10]. More precisely,  $(4')$  implies any two hyperbolic periodic saddles  $p, q \in \Lambda_g(U)$  have the same index (see the proof of Lemma 2.2 in [10]) and the later implies all periodic points in  $\Lambda_g(U)$  are hyperbolic (see Lemma 2.4 in [10]).  $(6) \Rightarrow (1')$ ,  $(a') \Rightarrow (a)$  ( $a = 1, 2, 3, 4$ ) and  $(2) \Rightarrow (3)$  are obvious and thus it is enough to show  $(1) \Rightarrow (2)$ ,  $(3) \Rightarrow (4)$  and  $(4) \Rightarrow (5)$ .

## 2 Proof of our main theorem

To prove  $(1) \Rightarrow (2)$ ,  $(3) \Rightarrow (4)$  and  $(4) \Rightarrow (5)$  in our main theorem, we divide into three lemmas. Firstly, we show a general lemma which implies  $(1) \Rightarrow (2)$ .

**Lemma 2.1.** *Let  $f : M \rightarrow M$  be a homeomorphism on a compact metric space  $M$  and let  $\Lambda$  be a transitive  $f$ –invariant subset. If  $f|_{\Lambda}$  satisfies the shadowing property, then  $\Lambda$  satisfies the transitive specification property.*

Remark: If we assume  $\Lambda$  is mixing in Lemma 2.1, then  $(f, \Lambda)$  satisfies the mixing specification property of Sigmund[11].

**Proof of Lemma 2.1** For any  $\varepsilon > 0$ , by shadowing property there exists  $\delta > 0$  such that any  $\delta$ –pseudo orbit in  $\Lambda$  can be  $\varepsilon$  shadowed by a true orbit in  $\Lambda$ .

Take and fix for  $\Lambda$  a finite cover  $\alpha = \{U_1, U_2, \dots, U_{r_0}\}$  by nonempty open balls  $U_i$  in  $\Lambda$  satisfying  $\text{diam}(U_i) < \delta$ ,  $i = 1, 2, \dots, r_0$ . Since  $\Lambda$  is transitive, for any  $i, j = 1, 2, \dots, r_0$ , there exist a positive integer  $X_{i,j}^{(1)}$  such that

$$f^{-X_{i,j}^{(1)}}(U_i) \cap U_j \neq \emptyset.$$

Let

$$M_1 = \max_{1 \leq i \neq j \leq r_0} X_{i,j}^{(1)}.$$

Similarly for any  $i, j = 1, 2, \dots, r_0$ , we can take a positive integer  $X_{i,j}^{(2)} \geq M_1$  such that

$$f^{-X_{i,j}^{(2)}}(U_i) \cap U_j \neq \emptyset.$$

Let

$$M_2 = \max_{1 \leq i \neq j \leq r_0} X_{i,j}^{(2)}.$$

By induction for any  $i, j = 1, 2, \dots, r_0$ , there is a sequence of increasing integers  $1 \leq X_{i,j}^{(1)} < X_{i,j}^{(2)} < \dots < X_{i,j}^{(n)} < \dots < +\infty$  such that

$$f^{-X_{i,j}^{(n)}}(U_i) \cap U_j \neq \emptyset$$

and

$$X_{i,j}^{(n)} \geq M_{n-1},$$

where

$$M_{n-1} = \max_{1 \leq i \neq j \leq r_0} X_{i,j}^{(n-1)}.$$

Setting  $M_0 = 0$ , clearly  $\{M_n\}_{n \geq 0}$  is an increasing sequence towards to  $+\infty$ .

Now let us consider a given sequence of points  $x_1, x_2, \dots, x_k \in \Lambda$ , and a sequence of positive numbers  $n_1, n_2, \dots, n_k$ . Take and fix  $U_{i_0}, U_{i_1} \in \alpha$  so that  $x_i \in U_{i_0}, f^{n_i}(x_i) \in U_{i_1}, i = 1, 2, \dots, k$ . Fixing an integer  $n \geq 1$ , take  $y_i \in U_{i_1}$  such that  $f^{X_{(i+1)_0, i_1}^{(n)}}(y_i) \in U_{(i+1)_0}$  for  $i = 1, 2, \dots, k-1$ . Take  $y_k \in U_{k_1}$  such that  $f^{X_{(k+1)_0, k_1}^{(n)}}(y_k) \in U_{1_0}$ . Thus we get a periodic  $\delta$ -pseudoorbit in  $\Lambda$ :

$$\{f^t(x_1)\}_{t=0}^{n_1} \cup \{f^t(y_1)\}_{t=0}^{X_{2_0, 1_1}^{(n)}} \cup \{f^t(x_2)\}_{t=0}^{n_2} \cup \dots \cup \{f^t(x_k)\}_{t=0}^{n_k} \cup \{f^t(y_k)\}_{t=0}^{X_{(k+1)_0, k_1}^{(n)}}.$$

Hence there exists a point  $z \in \Lambda$   $\varepsilon$ -shadowing the above sequence. More precisely,

$$d(f^{c_{i-1}+j}(z), f^j(x_i)) < \varepsilon, \quad j = 0, 1, \dots, n_i, \quad i = 1, 2, \dots, k,$$

where  $c_i$  is defined as follows:

$$c_i = \begin{cases} 0, & \text{for } i = 0 \\ \sum_{j=1}^i [n_j + X_{(j+1)_0, j_1}^{(n)}], & \text{for } i = 1, 2, \dots, k. \end{cases}$$

□

Secondly, we prove a lemma about the relationship of homoclinic related property and barycenter property, which deduces (3)  $\Leftrightarrow$  (4) of our main theorem.

**Lemma 2.2.** *Let  $f : M \rightarrow M$  be a diffeomorphism on a compact manifold  $M$ . Then for two hyperbolic periodic points  $p, q \in P(f)$ ,  $p, q$  have the barycenter property  $\Leftrightarrow W^u(\mathcal{O}(p)) \cap W^s(\mathcal{O}(q)) \neq \emptyset$ .*

### Proof of Lemma 2.2

”  $\Rightarrow$  ” : Let  $\varepsilon(p)$  and  $\varepsilon(q) > 0$  be as before with respect to  $p$  and  $q$ . Take  $\varepsilon = \min\{\varepsilon(p), \varepsilon(q)\}$ , and let  $N = N(\varepsilon, p, q) > 0$  be the number of barycenter property. For any  $n \geq 0$ , by barycenter property there is  $z_n \in \Lambda$  and an integer  $X_n \in [0, N]$  such that (i).  $d(f^j(z_n), f^j(p)) \leq \varepsilon$  for  $-n \leq j \leq 0$ ,  
(ii).  $d(f^{j+X_n}(z_n), f^j(q)) \leq \varepsilon$  for  $0 \leq j \leq n$ .

Take a subsequence  $\{n_k\}$  such that  $n_k \rightarrow \infty$  and  $X_{n_k} \equiv X$  for some fixed integer  $X \in [0, N]$ . Let  $z = \lim_{k \rightarrow +\infty} z_{n_k}$  by taking a subsequence again if necessary. By (i) and (ii) one has  $z \in W_{\varepsilon(p)}^u(p) \subseteq W^u(p)$  and  $f^X(z) \in W_{\varepsilon(q)}^s(q) \subseteq W^u(q)$ .

”  $\Leftarrow$  ” : If  $W^u(\mathcal{O}(p)) \cap W^s(\mathcal{O}(q)) \neq \emptyset$ , then we can take  $z \in W^u(\mathcal{O}(p)) \cap W^s(\mathcal{O}(q))$ . So for any  $\varepsilon > 0$ , there is  $N_1 = N_1(\varepsilon, p, q) > 0$  such that  $d(f^j(z), f^j(p)) < \varepsilon$  for all  $j \leq -N_1$  and  $d(f^j(z), f^j(q)) < \varepsilon$  for all  $j \geq N_1$ . Moreover, we can assume  $N_1$  to be a common multiple of the period of  $p$  and  $q$ . Put  $x = f^{-N_1}(z)$  and let  $N = 2N_1$ . Then for any two integers  $n_1, n_2, x$  is needed for barycenter property, i.e.,  $d(f^j(x), f^j(p)) = d(f^{j-N_1}(z), f^{j-N_1}(p)) < \varepsilon$  for all  $-n_1 \leq j \leq 0$  and  $d(f^{j+N}(x), f^j(q)) = d(f^{j+N_1}(z), f^{j+N_1}(q)) < \varepsilon$  for all  $0 \leq j \leq n_2$ .  $\square$

Before proving (4)  $\Rightarrow$  (5), we state a lemma which says that (4)  $\Rightarrow$  all hyperbolic points have the same index. A diffeomorphism  $f$  is said to be *Kupka – Smale* if the periodic points of  $f$  are hyperbolic and for any two periodic points  $p, q$  of  $f$ ,  $W^s(p)$  is transversal to  $W^u(q)$ . It is well known that the set of Kupka-Smale diffeomorphisms is  $C^1$ -residual in  $\text{Diff}(M)$  (see[9]). Note that if  $\mathcal{U}$  is an open set of  $\text{Diff}(M)$ , then the set of Kupka-Smale diffeomorphisms restricted in  $\mathcal{U}$  is still  $C^1$ -residual in  $\mathcal{U}$ .

**Lemma 2.3.** *Let  $f : M \rightarrow M$  be a diffeomorphism on a compact manifold  $M$ . Then condition (4) in Theorem 1.1 implies that for any two hyperbolic saddles  $p, q \in \Lambda_g(U) \cap P(g)$  with respect to  $g \in \mathcal{U}(f)$ ,  $\text{index}(p) = \text{index}(q)$ .*

**Proof** This proof is an adaption of Lemma 2.2 in [10]. Let  $\mathcal{U}(f)$  be as in condition (4) of Theorem 1.1. Fix a  $g \in \mathcal{U}(f)$ , and let  $p, q \in \Lambda_g(U) \cap P(g)$  be hyperbolic saddles. Then there is a  $C^1$ -neighborhood  $\mathcal{V}(g) \subseteq \mathcal{U}(f)$  such that for any  $\varphi \in \mathcal{V}(g)$ , there is continuations  $p_\varphi$  and  $q_\varphi$  (of  $p$  and  $q$ ) in  $\Lambda_\varphi(U)$ , respectively (Since  $\Lambda_\varphi(U) = \Lambda \subset \text{int}U$ , we can assume that  $\Lambda_g(U) \subset \text{int}U$  for any  $g \in \mathcal{U}(f)$  reducing  $\mathcal{U}(f)$  if necessary).

By contradiction, if  $\text{index}(p) < \text{index}(q)$  (the other case is similar), then we have

$$\dim W^s(p, g) + \dim W^u(q, g) < \dim M,$$

where  $W^s(p, g)$  and  $W^u(q, g)$  are the stable and unstable manifold of  $p$  and  $q$  with respect to  $g$ . Since the intersection of two residual sets is still residual, then the set of diffeomorphisms restricted in  $\mathcal{V}(g)$  satisfying not only Kupka-Smale but also condition (4) of Theorem 1.1 is still residual in  $\mathcal{V}(g)$ . Take such a diffeomorphism  $\varphi \in \mathcal{V}(g)$ . Then

$$W^s(p_\varphi, \varphi) \cap W^u(q_\varphi, \varphi) = \emptyset,$$

since  $\dim W^s(p, g) = \dim W^s(p_\varphi, \varphi)$  and  $\dim W^u(q, g) = \dim W^u(q_\varphi, \varphi)$ . On the other hand, since  $\varphi$  is a diffeomorphism satisfying condition (4) of Theorem 1.1, then

$$W^s(p_\varphi, \varphi) \cap W^u(q_\varphi, \varphi) \neq \emptyset.$$

This is a contradiction.  $\square$

**End of proof of (4) $\Rightarrow$ (5):** By lemma 2.3 we only need to prove that every periodic point  $p \in \Lambda_g(U)$  of  $g \in \mathcal{U}(f)$  is hyperbolic. By contradiction, suppose that  $p \in \Lambda_g(U)$  of  $g \in \mathcal{U}(f)$  is not hyperbolic. Then by Lemma 2.4 in [10], there is  $\varphi \in \mathcal{U}(f)$  possessing hyperbolic points  $q_1$  and  $q_2$  in  $\Lambda_\varphi(U)$  with different indices. This is a contradiction to Lemma 2.3.  $\square$

### 3 One remark for volume-preserving version.

Let  $\omega$  be a volume measure on the smooth compact manifold  $M$  and  $\text{Diff}_\omega(M)$  be the space of diffeomorphisms preserving  $\omega$ . We point out the statements in Theorem 1.1 can be changed for the volume-preserving diffeomorphisms, since all the main techniques can be replaced by the ones of volume-preserving version. I.e., the set of transitive volume-preserving Anosov diffeomorphisms is a characterization of the set of volume-preserving diffeomorphisms satisfying  $C^1$ -stable (or generic-stable) shadowable property,  $C^1$ -stable (or generic-stable) transitive or mixing specification property or  $C^1$ -stable (or generic-stable) barycenter property. Let's explain it more precisely as follows. The equivalence of condition (5) and (6) (see [7]) in Theorem 1.1 can be replaced by the recent result in [2], Frank's lemma[5] (important to prove Lemma 2.4 in [10] which is needed in our proof of (4) $\Rightarrow$ (5)) can be replaced by the pasting lemma for volume-preserving systems(see [3]) and Kupka-Smale property for volume-preserving case can be found in [8]. In particular, we point out that we need not assume the robust transitivity of  $M$  when we prove the result that one volume-preserving diffeomorphism satisfying  $C^1$ -stable (or generic-stable) shadowable property is Anosov, since generic volume-preserving diffeomorphisms are transitive from [4](and so that by Lemma 2.1 generic-stable shadowing implies generic-stable transitive specification for volume-preserving). Moreover, we note that volume-preserving Anosov diffeomorphisms are always transitive from the viewpoint of structurally stable property of Anosov systems and the transitivity of generic volume-preserving diffeomorphisms[4], since every volume-preserving Anosov diffeomorphism has a topologically conjugated diffeomorphism arbitrarily nearby which can also be chosen transitive from [4] and transitivity is an invariant property under conjugation. So the statements above for volume-preserving case can be directly as a characterization of (not necessarily adding "transitive") volume-preserving Anosov diffeomorphisms.

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